tion eliminates the control and observation spillovers but has dynamic spillover. The trailing zeros in the force distribution and measurement distribution matrices eliminate the control and observation spillovers. Dynamic spillover is due to the nonzero super- and subdiagonal submatrices in the mass matrix. For dynamic response simulation, the block-tridiagonal form can lead to an efficient time-step solution and can save storage space.[2]

Summary

A Krylov model reduction algorithm for an undamped structural dynamics system is presented. The reduced-order model obtained matches low-frequency moments. The transformed system equation in Krylov coordinates reflects the structure of a tandem system.

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Optimal Feedback Gains for Three-Dimensional Large Angle Slewing of Spacecraft

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Introduction

THREE-DIMENSIONAL, large angle slewing or rotational maneuvers occur during retargeting of spacecraft or spacecraft appendages. This is a nonlinear control problem that has received considerable attention in the recent literature (see, e.g., Refs. 1–5). In a representative work, Junkins and Turner[1] consider various methods of optimal control of slewing of rigid and flexible bodies and discuss numerical techniques of solving the associated two-point boundary value problem to obtain open-loop control profiles. A simpler approach is taken by Wie and Barba,[5] who propose feedback control in terms of Euler parameters and angular velocity components and show the effectiveness of such control. Because the dynamics are strongly nonlinear, selection of the feedback gains in this approach is based on good engineering judgment and verified by simulation. This paper provides a systematic procedure for determining these feedback control gains based on optimal control theory[7,8] and an application of a state-of-the-art optimization technique.[9] The method is applied to a problem in which a rigid body is simultaneously rotated in three axes to capture a desired orientation using feedback control torques that have saturation constraints.

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Statement of the Problem

The problem concerns reorienting a rigid body from its current orientation at rest to a desired orientation, also at rest, in a given time in an optimal manner that is to be defined. Let x_1, x_2, x_3, and x_4 denote the four Euler parameters describing the current orientation of a rigid body undergoing large three-dimensional rotation. Kinematical differential equations for Euler parameters are given by[6]

$$\dot{x}_1 = 0.5\ (x_5 x_4 - x_6 x_3 + x_7 x_2) \tag{1a}$$

$$\dot{x}_2 = 0.5\ (x_5 x_3 + x_6 x_4 - x_7 x_1) \tag{1b}$$

$$\dot{x}_3 = 0.5\ (-x_5 x_2 - x_6 x_1 + x_7 x_4) \tag{1c}$$

$$\dot{x}_4 = -0.5\ (x_5 x_1 + x_6 x_2 + x_7 x_3) \tag{1d}$$

where x_5, x_6, and x_7 are the angular velocity components along the body axes. The dynamical equations are given by the following Euler equations:

$$\dot{x}_5 = [T_1 - (I_3 - I_2)x_6 x_7]/I_1 \tag{2a}$$

$$\dot{x}_6 = [T_2 - (I_1 - I_3)x_5 x_7]/I_2 \tag{2b}$$

$$\dot{x}_7 = [T_3 - (I_2 - I_1)x_5 x_6]/I_3 \tag{2c}$$

where T_i and I_i are the ith body axis component of control torque and the centroidal principal moment of inertia for the body, respectively. Feedback control is proposed in the manner of Ref. 5, but with the extra constraint of torque saturation as follows:

$$\begin{aligned} T_1 &= -k_1 e_1 - k_2 x_5 \quad \text{if} \quad |T_1| < T_s \\ &= T_s \operatorname{sgn}(-k_1 e_1 - k_2 x_5) \qquad \text{otherwise} \end{aligned} \tag{3a}$$

$$\begin{aligned} T_2 &= -k_3 e_2 - k_4 x_6 \quad \text{if} \quad |T_2| < T_s \\ &= T_s \operatorname{sgn}(-k_3 e_2 - k_4 x_6) \qquad \text{otherwise} \end{aligned} \tag{3b}$$

$$\begin{aligned} T_3 &= -k_5 e_3 - k_6 x_7 \text{ if } |T_3| < T_s \\ &= T_s \operatorname{sgn}(-k_5 e_3 - k_6 x_7) \qquad \text{otherwise} \end{aligned} \tag{3c}$$

where e_1, e_2, and e_3 are the components of the Euler vector for error in orientation given by the Euler parameters representing the single-axis rotation that it takes to go from the current attitude to the reference attitude. These are computed according to the rules of quaternion algebra[6]

$$e_1 = r_4 x_1 + r_3 x_2 - r_2 x_3 - r_1 x_4 \tag{4a}$$

$$e_2 = -r_3 x_1 + r_4 x_2 + r_1 x_3 - r_2 x_4 \tag{4b}$$

$$e_3 = r_2 x_1 - r_1 x_2 + r_4 x_3 - r_3 x_4 \tag{4c}$$

where r_1, r_2, r_3, and r_4 are the Euler parameters for the desired reference orientation. Now, the optimal control problem can be stated as follows. Find the feedback control gains k_1, k_2, k_3, k_4, k_5, and k_6 in Eqs. (3) so as to minimize the performance criterion J,

$$J = W_1 e_1^2(t_f) + W_2 e_2^2(t_f) + W_3 e_3^2(t_f) + \int_0^{t_f} [e_1^2 + e_2^2 + e_3^2 + R_1 T_1^2 + R_2 T_2^2 + R_3 T_3^2] \, dt \tag{5}$$

Choice of the performance criterion is arbitrary, and depending on the situation, Eq. (5) may be modified to include penalties on angular velocity components and even torque rate inputs, the latter requiring the addition of new state equations.

Optimal Control Formulation

Let us introduce a state variable x_8 in addition to those already introduced in Eqs. (1) and (2) to set up the following differential equation:

$$\dot{x}_8 = e_1^2 + e_2^2 + e_3^2 + R_1 T_1^2 + R_2 T_2^2 + R_3 T_3^2, \qquad x_8(0) = 0 \tag{6}$$

Then the performance criterion of Eq. (5) reduces to one stated exclusively in terms of the conditions at the given final time t_f

$$J = W_1 e_1^2(t_f) + W_2 e_2^2(t_f) + W_3 e_3^2(t_f) + x_8(t_f) \tag{7}$$

Minimizing the functional J in the space of the input control parameters ($k_1, \ldots, k_6$ in this case) is a problem in the calculus of variations.[7,8] The required algorithm can be summarized as follows.

1) Write the state differential equations, Eqs. (1–4) and (6), and the cost function, Eq. (7), as

$$\dot{x} = f(x, k), \qquad x(0) = x_0 \tag{8a}$$

$$J = \phi[x(t_f), k] \tag{8b}$$

where x is the state vector and k is the gain (parameter) vector.

2) Introduce the Hamiltonian, defined in terms of the state vector x and adjoint variables vector λ, as

$$H(x, k, \lambda) = f^T(x, k)\lambda \tag{9}$$

3) The differential equations in the adjoint variable are

$$\dot{\lambda} = -\frac{\partial H(x,k,\lambda)}{\partial x} = -\frac{\partial f^T(x,k)}{\partial x}\lambda \tag{10a}$$

$$\lambda(t_f) = \frac{\partial \phi[x(t_f), k]}{\partial x(t_f)} \tag{10b}$$

4) The gradient of the cost function with respect to the control gains is given by the key relation[8]

$$\frac{\partial J}{\partial k} = \frac{\partial \int_0^{t_f} H(x,k,\lambda) \, dt}{\partial k} = \int_0^{t_f} \frac{\partial f^T(x,k)}{\partial k} \lambda \, dt \tag{11}$$

For optimality, the gradient should be zero. Note that the execution of the algorithm requires that Eq. (8) be integrated forward in time, Eq. (9) be integrated backward in time, and the two sets of time histories stored to form the integrand in Eq. (11). To accommodate control saturation in this formulation, the following transparent relations are used in computing the various gradients of $f(x,k)$ needed in Eqs. (10) and (11) whenever saturation occurs in any control torque:

$$\frac{\partial T_i(x,k)}{\partial x} = \frac{\partial T_i(x,k)}{\partial k} = 0 \quad \text{if} \quad |T_i| = 0, \quad i = 1,2,3 \tag{12}$$

Numerical Optimization

With the cost function and its gradient with respect to the parameter vector (control gains) computed in Eqs. (7) and (11), respectively, the stage is set for applying numerical methods of optimization. One of the mathematically most sophisticated and rapidly convergent techniques that have been available for some time is that due to Broydon, Fletcher, Goldfarb, and Shanno (BFGS).[9] The BFGS algorithm is a first-order method in that it only requires gradient information, but it has quadratic convergence properties similar to Newton's method. The BFGS algorithm with line search is summarized for the ith iteration for the gain vector k^i as[9]

$$i = 0; \qquad H^i = I, \qquad k^i \text{ assumed} \tag{13a}$$

$$S^i = -H^i \nabla J(k^i) \tag{13b}$$

$$k^{i+1} = k^i + \alpha_i^* S^i \tag{13c}$$

where I is the identity matrix, ∇J is the gradient of J and α_i^* the minimum along S^i. H^i approximates the inverse Hessian and is updated in terms of a few intermediate steps:

$$p = k^i - k^{i-1} \tag{14a}$$

$$y = \nabla J(k^i) - \nabla J(k^{i-1}) \tag{14b}$$

$$\sigma = p^T y \tag{14c}$$

$$\tau = y^T H^i y \tag{14d}$$

$$D^i = \frac{\sigma + \tau}{\sigma^2} p\, p^T - \frac{1}{\sigma}[H^i y\, p^T - p\,(H^i y)^T] \tag{14e}$$

$$H^{i+1} = H^i + D^i \tag{14f}$$

An appropriate convergence criterion is used to terminate the iterations.

Simulation Results

The numerical problem solved is as follows: a rigid body with centroidal principal inertias $I_1 = 16{,}640$ kg-m^2, $I_2 = 11{,}660$ kg-m^2, and $I_3 = 16{,}590$ kg-m^2 is to be reoriented in 30 s so as to capture a desired attitude that requires a simultaneous three-axis rotation of 160 deg in yaw, 20 deg in pitch, and -20 deg in roll. It is assumed that thrusters are available with throttling for torque-shaping, with torque saturation occurring at 900 N-m. The numerical simulation starts by computing the body three direction cosine matrix for the desired orientation and then backing out the reference (step-input) Euler parameters from this direction cosine matrix.[6]

The actual numerical optimization, employing the BFGS scheme with line search by the method of golden sections, is done by the publicly available code ADS,[10] which is used as a driver for the routine that computes the analytic gradients. The latter performs the forward integration of Eqs. (8), the backward integration of Eqs. (10), and the integral evaluation of Eq. (11). ADS has an option of computing the necessary gradients by finite differencing, and this is used as a "sanity check" of the computed analytic gradients. The results presented in Figs. 1 and 2 were obtained identically by the two methods of computing gradients, with the analytic gradients method requiring 70 function evaluations and the numerical differentiation method requiring 470 function evaluations before the optimization process converged in each case. The results shown are for the following values of the weighting

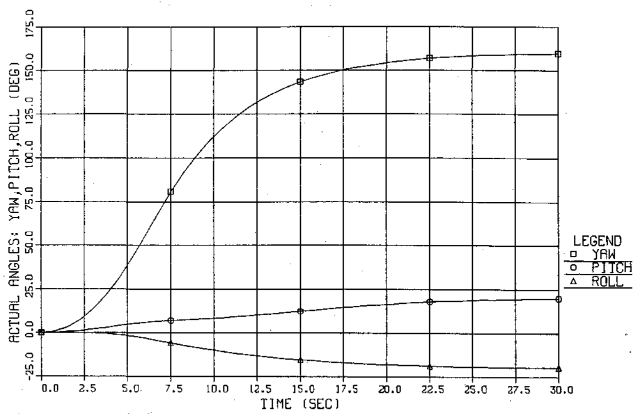

Fig. 1 **Three-dimensional control of rigid spacecraft: yaw = 160 deg; pitch = 20 deg; and roll = − 20 deg.**

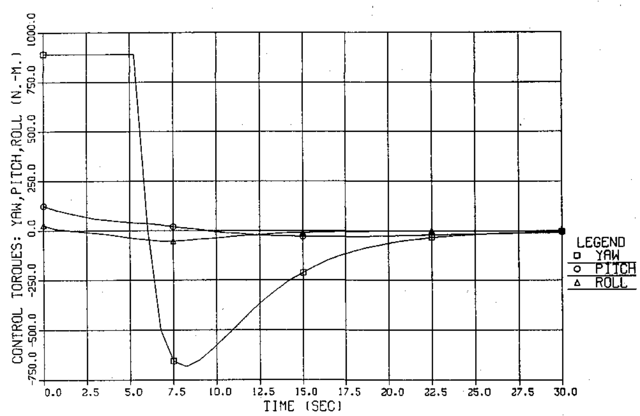

Fig. 2 **Three-dimensional slewing control of rigid spacecraft: feedback control torques in yaw, pitch, and roll.**

Table 1 Initial guess and converged values of gains and final values of gradient of cost function

Initial k	1.825×10^4	2.464×10^4	1.279×10^4	1.727×10^4	1.819×10^4	2.457×10^4
Final k	1.149×10^4	2.786×10^4	6.334×10^2	3.118×10^3	1.908×10^2	3.169×10^4
$\partial J/\partial k$	-0.82×10^{-4}	0.43×10^{-4}	0.18×10^{-3}	0.12×10^{-5}	0.10×10^{-4}	-0.76×10^{-7}

parameters in Eq. (5): $W_1 = W_2 = W_3 = 2.0 \times 10^3$, $R_1 = 10^{-7}$, $R_2 = R_3 = 10^{-5}$. Figure 1 shows that the desired attitude is captured with little overshoot and with essentially zero terminal angular velocity. The corresponding control torque time histories are given in Fig. 2, showing initial saturation in the yaw torque and the required torque shaping. Table 1 shows the initial guess and final value of the gains, along with the final value of the gradients. It shows that the initial guesses, based on settling time considerations for uncoupled linear equations, were *not* optimal, whereas the gains computed by the procedure given in this paper, while keeping the system stable as indicated by the slewing response, also provided optimal performance.

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Statistical Linearization for Multi-Input/Multi-Output Nonlinearities

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I. Introduction

COVARIANCE analysis provides an alternative to Monte Carlo simulation for evaluating the performance of interconnected nonlinear dynamical systems under noisy environments and, in many practical situations, is more efficient than Monte Carlo simulation.[1,2] Applications of covariance analysis to such nonlinear systems as guidance filters for tactical missiles and space interceptor-target engagement have been reported.[1,3]

A crucial step in the covariance analysis algorithm is to obtain random-input describing functions, i.e., the linear

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